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SOME NOTES ON ALOHA CHANNELS WITH CAPTURE

1. INTRODUCTION

Radio receivers which receive new and retransmitted packets at times that form a Poisson point process are considered; all packets have the same length. Initially, a single receiver with perfect capture is considered. It is found that the normalized "ALOHA capacity" for such a system is 1/2 for an unslotted system and one for a slotted system. Since these capacities are independent of the spatial distribution of the throughput, the Sisyphus distance is a function of the throughput distribution but not the ALOHA channel. Thus optimal repeater spacings in a net would appear to be primarily a function of the distribution of users.

Some simple examples are used to show that imperfect capture does not change these capacities -- in theory. Finally a very simplified model with two terminals and perfect capture indicates that an arbitrary spatial distribution of throughput will be distributed between the two terminals so that each operates at peak efficiency.

2. A SINGLE TERMINAL WITH PERFECT CAPTURE

The users are assumed to transmit and retransmit packets of normalized length 1. In addition, the location and available power of the users combined with the perfect capture is assumed to lead to a complete ordering of the users. Let this ordering be indexed by the positive capture index x. Thus if user i, i=1,2, has capture index x_i and $x_1 > x_2$, then the receiver hears only the transmission from user 2 whenever both users transmit simultaneously.

For a collection of users with equal transmitted power, the capture index can be taken to be r, the distance from the user to the receiver, if the received power decreases monotonically with distance; e.g., $1/r^2$ or $1/r^4$. For a collection of users with transmitted power proportional to r^4 and a square law

received power, the capture index can be taken to be $1/r$. However, for a collection of users with transmitted power proportional to r^4 and a fourth power received power, there is no capture index.

Define $s(x)$ as the normalized packet throughput per unit capture index at capture index x. For example, if the model of Abramson [1] is considered, the capture index is the radius r and the normalized packet throughput per unit capture index $s(r)$ is the product of $2\pi r$ and the normalized packet throughput per unit area of [1]. Define $g(x)$ as the normalized traffic per unit capture index at capture index x.

Now by making use of the usual Poisson and steady-state arguments, one finds

$$s(x) = g(x)e^{-\delta G(x)}, \tag{1a}$$

where the cumulative traffic at capture index x is

$$G(x) = \int_0^x g(y)dy \tag{1b}$$

and the interference window δ is one for a slotted system and two for an unslotted system. After integrating both sides of (1a) and evaluating the constant of integration at zero capture index, one finds the cumulative throughput at x

$$S(x) = \int_0^x s(y)dy \tag{1c}$$

is only a function of the cumulative traffic at x:

$$S(x) = [1-e^{-\delta G(x)}]/\delta \ . \tag{2a}$$

Since the total normalized packet throughput

$$S(\infty) = [1 - e^{-\delta G(\infty)}]/\delta \tag{2b}$$

it is obvious that by choosing the total traffic sufficiently large, the total throughput can be made arbitrarily close to $(1/\delta)$. Thus the "ALOHA capacity" of a system with perfect capture and identical packet lengths is one for a slotted system and 1/2 for an unslotted system. Note that this capacity is

independent of the actual distribution of throughput; it is not a spatial capacity. Also note that it is achieved only by having an infinite average traffic.

One can also express the cumulative traffic in terms of the cumulative throughput:

$$G(x) = -(1/\delta) \ln [1-\delta S(x)] \quad . \qquad (2c)$$

After differentiating (2c) and dividing by s(x), one finds that the normalized delay at capture index x,

$$d(x) = g(x)/s(x) \quad , \qquad (3a)$$

is only a function of the cumulative throughput at capture index x

$$d(x) = [1 - \delta S(x)]^{-1} \quad . \qquad (3b)$$

Figure 1 is a plot of the delay d(x) as a function of the cumulative throughput S(x).

Note that the delay becomes infinite at a capture index x_0 such that the cumulative throughput is equal to the capacity. In considering constant throughput per unit area, Abramson [1], refers to this value of the index as the Sisyphus distance. Thus the concept of a Sisyphus index x_0 can be generalized to an arbitrary distribution of throughput:

$$S(x_0) = 1/\delta \quad . \qquad (4)$$

Note that the Sisyphus index is a function of only the distribution of throughput; it is independent of the ALOHA channel. Thus, in designing nets with several repeaters, the spatial distribution of throughput is the primary factor determining the repeater spacings. Each user should be within the Sisyphus index of at least one repeater.

3. SOME EXAMPLES WITH IMPERFECT CAPTURE

In this section it is shown that imperfect capture does not change the "ALOHA capacity." This is accomplished by judiciously choosing the traffic

distribution and then computing the resultant throughput distribution. Although imperfect capture does not change the "ALOHA capacity" in theory; in practice it is an important phenomena that needs more study than it receives here.

For a collection of unit-length, new and retransmitted, packets that arrive at Poisson times, one finds that with imperfect capture the normalized throughput per unit capture index is

$$s(x) = g(x)e^{-\delta G(ax)}, \tag{5}$$

where the capture ratio parameter a is larger than one.

For the first example let the normalized traffic distribution per unit capture index be

$$g(x) = g_0 x^{n-1} \tag{6a}$$

where the parameters n and g_0 are larger than zero. Then, the throughput distribution is

$$s(x) = g_0 x^{n-1} e^{-\delta g_0 a^n x^n/n}; \tag{6b}$$

the delay is

$$d(x) = e^{\delta g_0 a^n x^n/n}; \tag{6c}$$

and the cumulative throughput is

$$S(x) = [1-e^{-\delta g_0 a^n x^n/n}]/[\delta a^n] . \tag{6d}$$

By choosing n sufficiently close to zero, the total throughput $S(\infty)=(\delta a^n)^{-1}$ can be made arbitrarily close to the "ALOHA capacity" for any value of a. Of course, the delay also becomes very large.

For the second example let the traffic distribution be an infinite number of impulses:

$$g(x) = \sum_{i=0}^{\infty} g_i \delta(x-x_i) . \tag{7a}$$

Furthermore, let the impulses be spaced such that

$$ax_i < x_{i+1}, \quad i=0,1,\ldots; \tag{7b}$$

it will be shown shortly that this is equivalent to spacing the users so that the imprefection of the capture has no effect. One then finds that the throughput with imperfect capture is identical to the throughput with perfect capture. Thus by choosing the total traffic, $G(\infty)$, sufficiently large, the total throughput $S(\infty)$ can be made arbitrarily close to the "ALOHA capture."

4. AN EXAMPLE WITH TWO TERMINALS

In this section an extremely simple example with two terminals, both with perfect capture, is considered. It is found that the "ALOHA capacity" of the system is twice that of a single terminal and is independent of the actual throughput distribution. Thus the two terminals automatically distribute the throughput so that each operates at peak efficiency.

The two terminals are located on the end points of a unit-length line segment; all the users are located on the line segment. Let $s(x)$ and $g(x)$ be the normalized throughput and traffic, respectively, per unit length at the point x on the line segment. Now if the speed of propagation is assumed to be infinite, all new and retransmitted packets are heard simultaneously, but at different power levels, at both terminals. If all the packets have normalized duration of unity, again, have equal transmitted power and arrive at Poisson times, one finds the normalized throughput at x, $s(x)$, is the product of the normalized traffic at x, $g(x)$, and the probability that a packet is successfully received at either, but not both, terminal:

$$s(x) = g(x)\{e^{-\delta G(x)} + e^{-\delta[G(1)-G(x)]} - 2e^{-\delta G(1)}\}. \tag{8a}$$

After integrating both sides of (8a) and evaluating the constant of integration at zero, one finds the cumulative throughput is a simple function of the total traffic $G(1)$ and the cumulative traffic $G(x)$:

$$S(x) = \{1-e^{-\delta G(1)}\}/\delta - \{e^{-\delta G(x)} - e^{-\delta[G(1)-G(x)]}\}/\delta - G(x)e^{-\delta G(1)} . \tag{8b}$$

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By evaluating the above equation at unity, one finds the total throughput, S(1), is a simple, but transcendental, function of the total traffic:

$$S(1) = 2\{1 - e^{-\delta G(1)}[1+\delta G(1)/2]\}/\delta \quad . \tag{8c}$$

Figure 2 is a plot of the total throughput as a function of the total traffic.

By introducing the relative traffic

$$G_0(x) = G(x) - \tfrac{1}{2} G(1) \tag{9a}$$

and the relative throughput

$$S_0(x) = S(x) - \tfrac{1}{2} S(1) \tag{9b}$$

and performing some algebraic gymnastics on (8b), one finds the relative throughput is a simple odd function of the relative traffic and the total traffic:

$$S_0(x) = \{2\, e^{-\delta G(1)/2} \sinh \delta G_0(x) - \delta G_0(x) e^{-\delta G(1)}\}/\delta \quad . \tag{9c}$$

Figure 3 is a plot of the relative throughput as a function of the total throughput for several values of the total traffic.

By differentiating both sides of (9c) with respect to x one finds

$$s(x) = g(x)\{2e^{-\delta G(1)/2} \cosh \delta G_0(x) - e^{-\delta G(1)}\} \quad . \tag{10a}$$

Now since G(1) is a function of S(1) and $G_0(x)$ is a function of S(1) and $S_0(x)$, the average delay encountered by a user at the point x is a function of only the relative throughput at x and the total throughput:

$$d(x) = \frac{g(x)}{s(x)} = [2e^{-\delta G(1)/2} \cosh \delta G_0(x) - e^{-\delta G(1)}]^{-1} \quad . \tag{10b}$$

Furthermore, since $S_0(x)$ is an odd function of $G_0(x)$ and d(x) is an even function of $G_0(x)$, the average delay is an even function of the relative throughput. Figure 4 is a plot of the average delay as a function of the relative throughput for several values of the total throughput.

Note that by choosing the total traffic sufficiently large, the total throughput can be made arbitrarily close to (2/δ) for any spatial distribution of throughput. Since (2/δ) is the sum of the "ALOHA capacities" of the individual terminals, the "ALOHA capacity" of these two terminal systems is twice that of a single terminal system. Furthermore, this capacity is independent of the actual spatial distribution of throughput; for any spatial distribution, the capture effect automatically balances the throughput so that each terminal operates at capacity. The balancing of the throughput between the two terminals for arbitrary spatial distributions of throughput is further evidenced by the symmetry of the delay and relative throughput equations.

5. EXTENSIONS

Certainly the most important extension of these results would be to include imperfect capture effects. A technique for extending the results of this note has not been found yet; however, by considering examples where the traffic distribution is specified and the resulting throughput is then computed it should be possible to determine the effects of imperfect capture.

Another important extension of these results would be to consider more general spatial configurations with multiple terminals. A technique for accomplishing this analysis, other than one similar to the brute force technique suggested for imperfect capture, is not known either.

An additional extension of these results that should be relatively simple to calculate (for terminals with perfect capture) would be to consider mixtures of packet sizes and reservations schemes as in [2]. Other effects to consider would be blocking [3], [4] and carrier sensing [5], [6], [7].

REFERENCES

[1] Abramson, Norman: "The Spatial Capacity of an ALOHA Channel," Packet Radio Temporary Note No. 49, NIC Document No. 16087, Stanford Research Institute, April 30, 1973.

[2] Gaarder, N. T.: "ARPANET Satellite System," ARPANET Satellite System Note No. 3, NIC Document No. 11285, Stanford Research Institute, April 24, 1972.

[3] Metcalfe, Robert: "Steady-State Analysis of a Slotted and Controlled ALOHA System with Blocking," ARPANET Satellite System Note No. 16, NIC Document No. 11624, Stanford Research Institute, October 18, 1972.

[4] Lu, Shyue-Ching: "Dynamic Analysis of Slotted ALOHA with Blocking," ARPANET Satellite System Note No. 36, NIC Document No. 14790, Stanford Research Institute, March 12, 1973.

[5] Binder, Richard: "Steady State Analysis of an ALOHA Carrier Sense," ALOHA System Working Document CCG/W-17, University of Hawaii, June 1973.

[6] Kleinrock, L. and Tobagi, F.: "Throughput in Carrier-Sense (Autoslot) Packet Radio Systems," Packet Radio Temporary Note No. 24, NIC Document No. 14212, Stanford Research Institute, February 12, 1973.

[7] Kleinrock, L. and Tobagi, F.: "Carrier-Sense with Initial Random Transmission Delay," Packet Radio Temporary Note No. 37, NIC Document No. 15[illegible], Stanford Research Institute, March 1973.

Figure 1 AVERAGE DELAY VS. CUMULATIVE THROUGHPUT

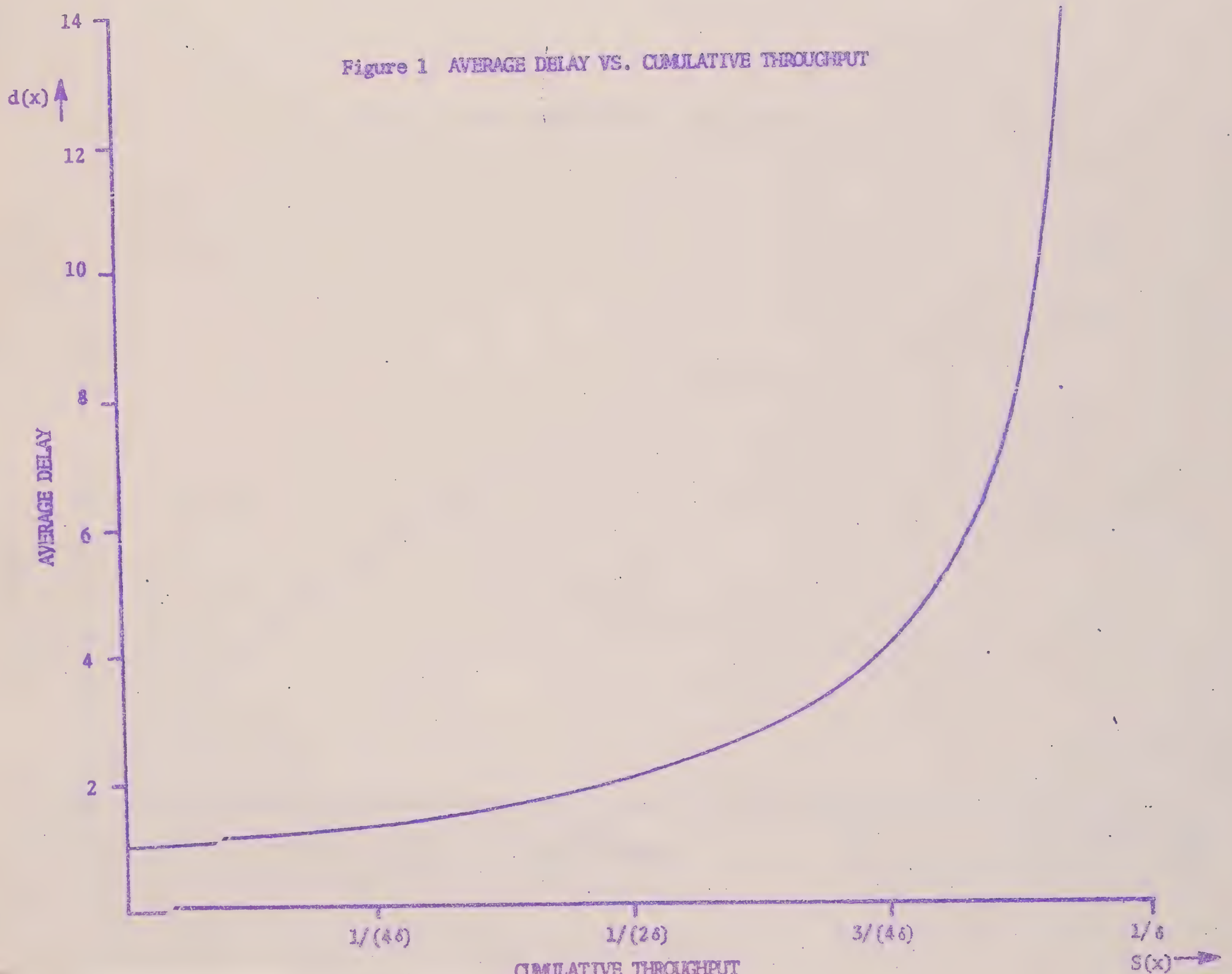
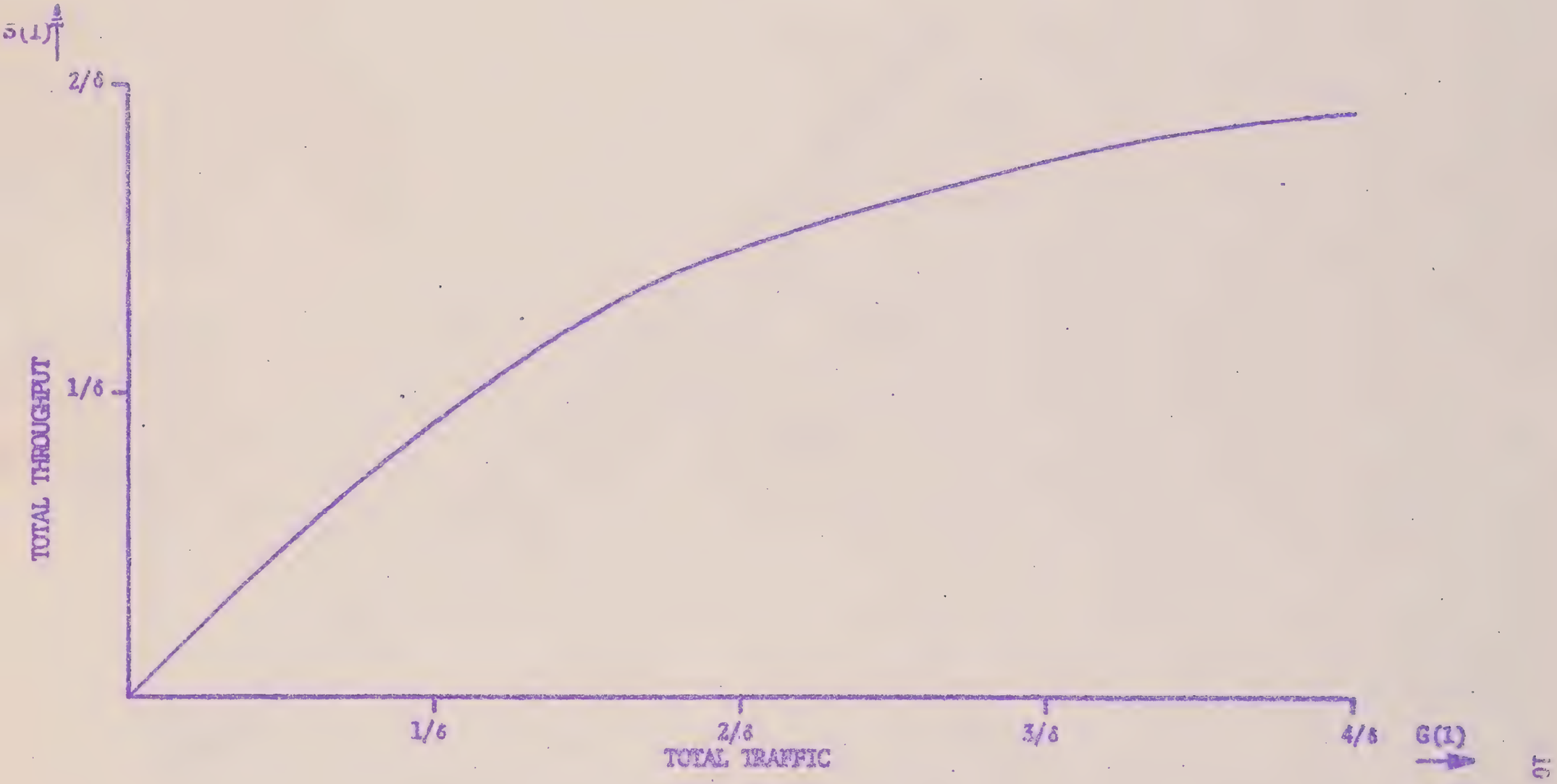


Figure 2 TOTAL THROUGHPUT VS. TOTAL TRAFFIC



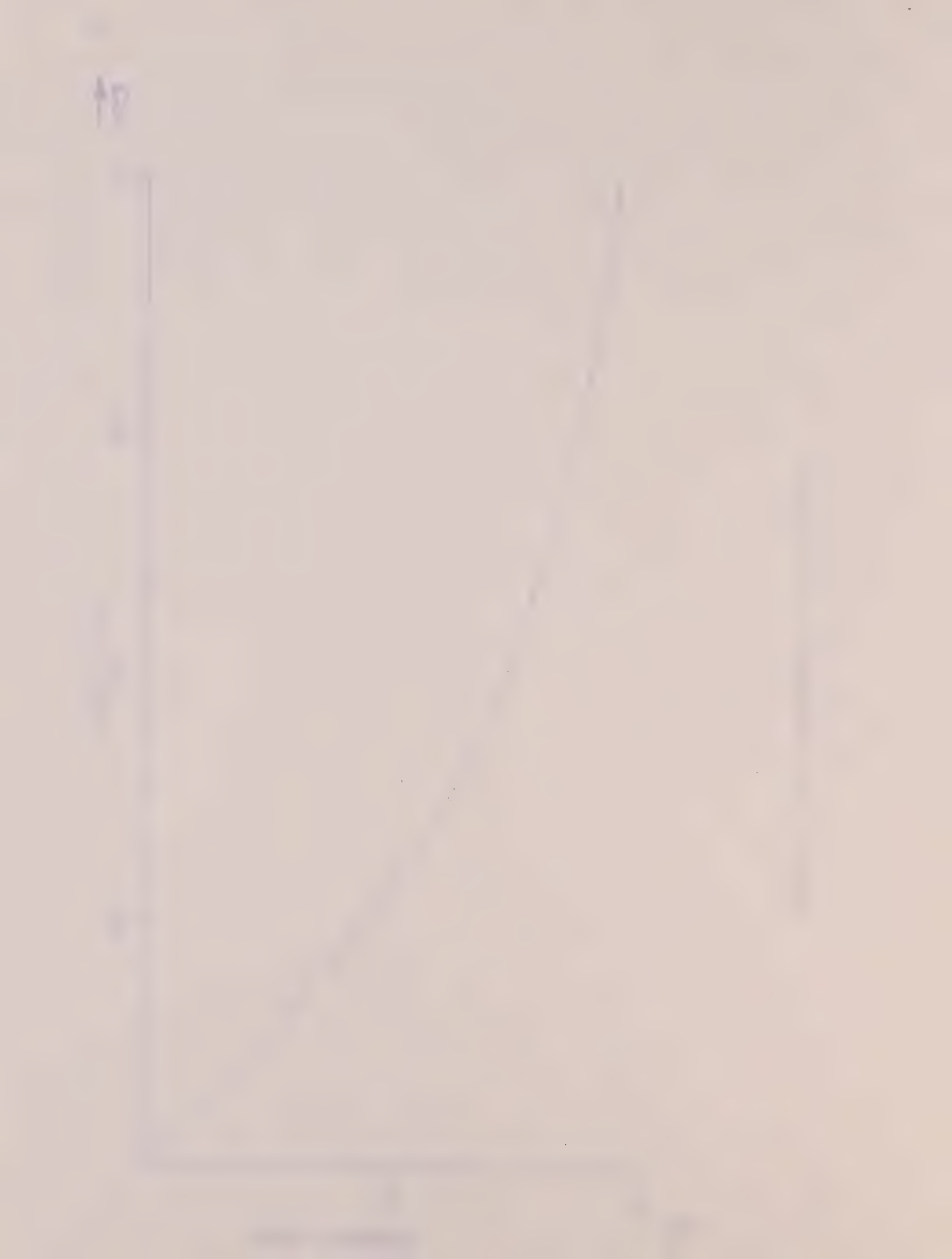
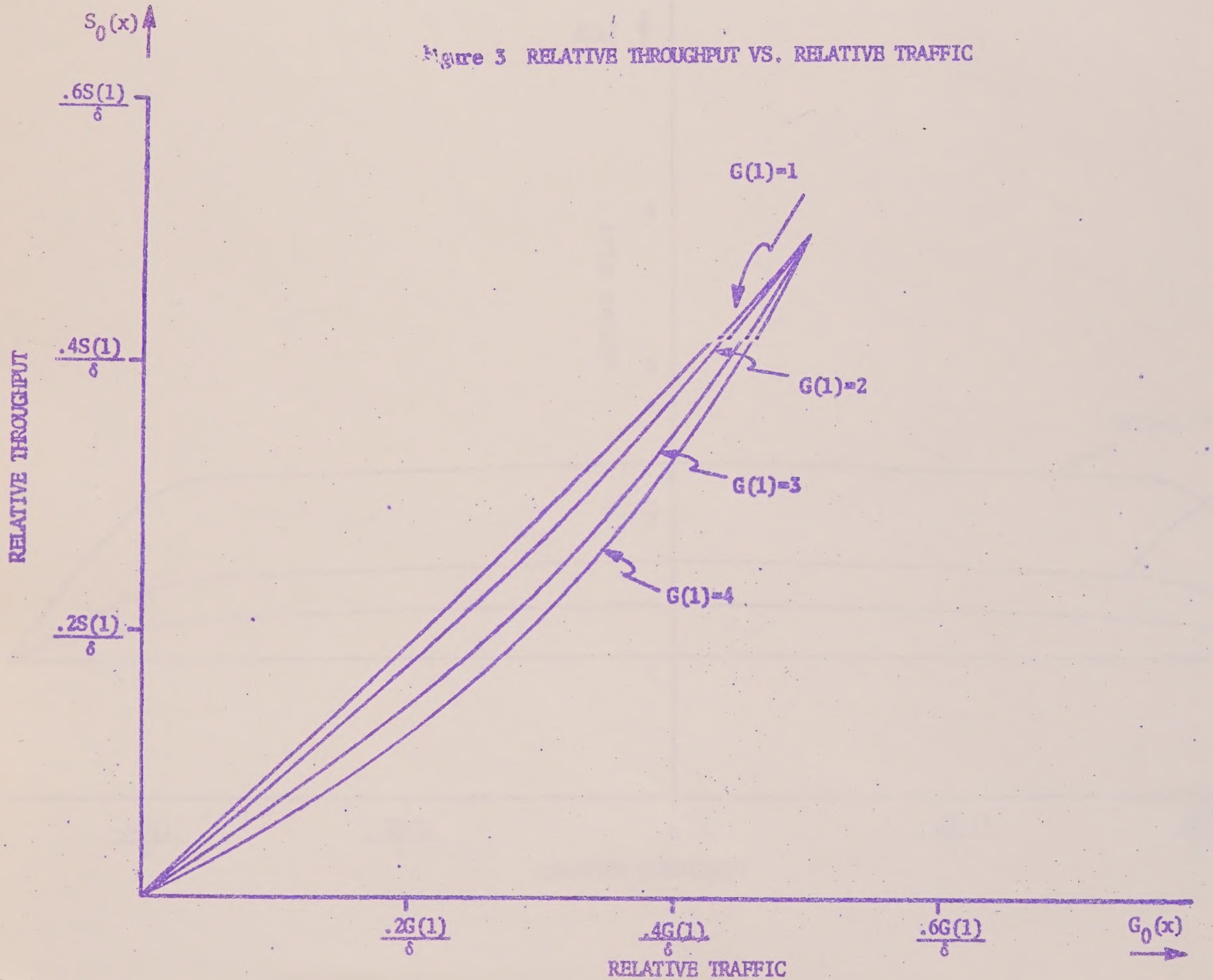


Figure 3 RELATIVE THROUGHPUT VS. RELATIVE TRAFFIC



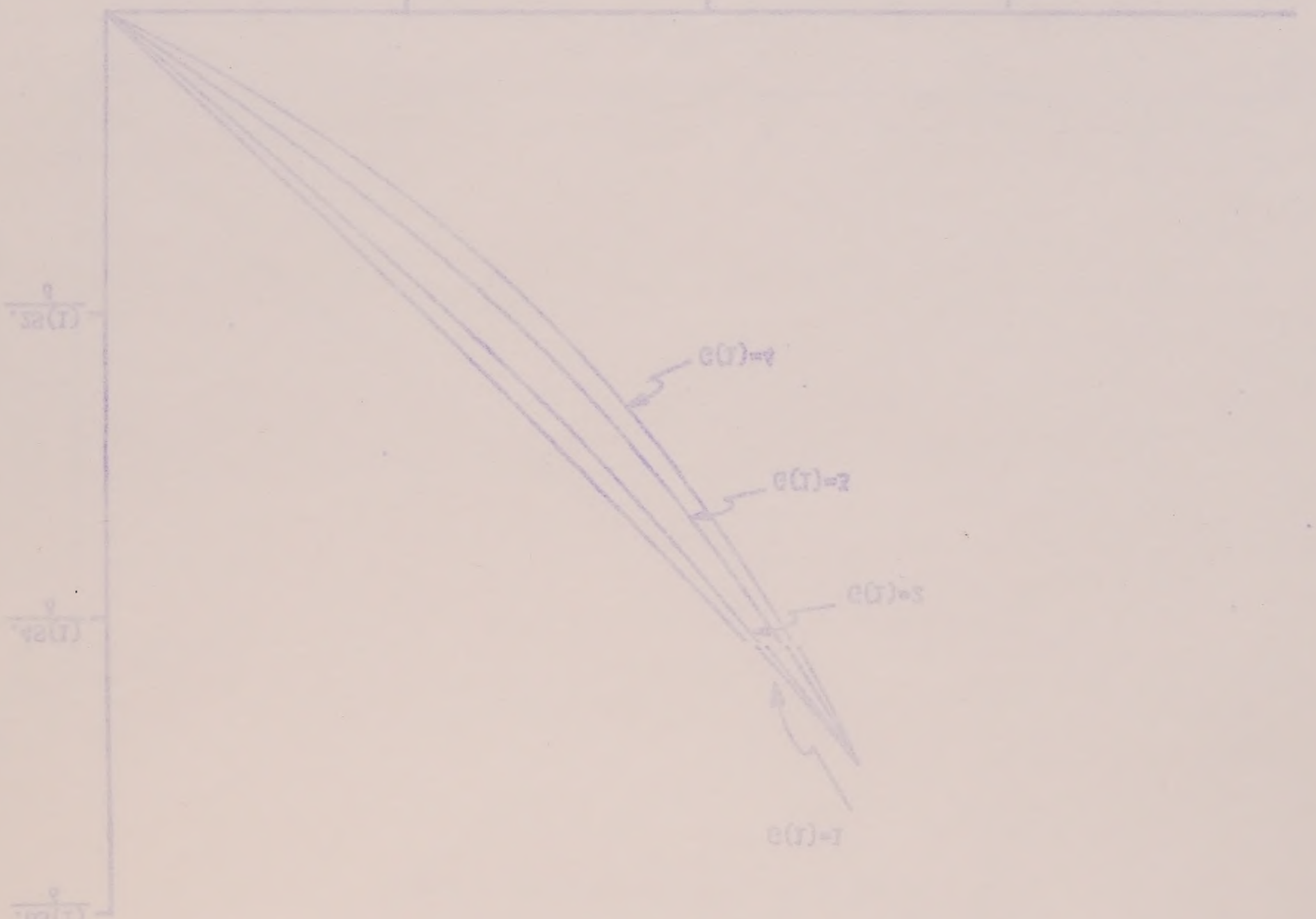
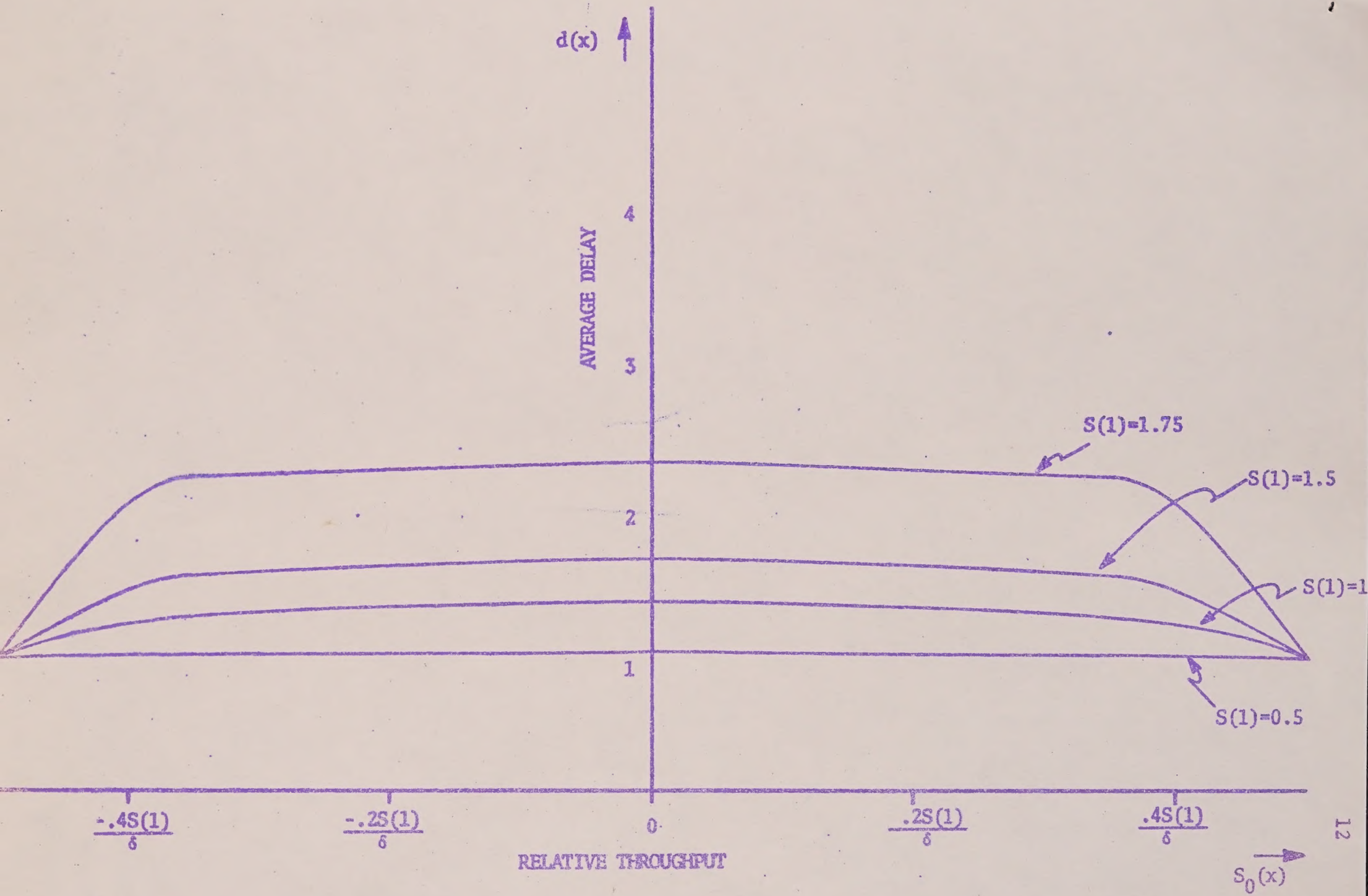
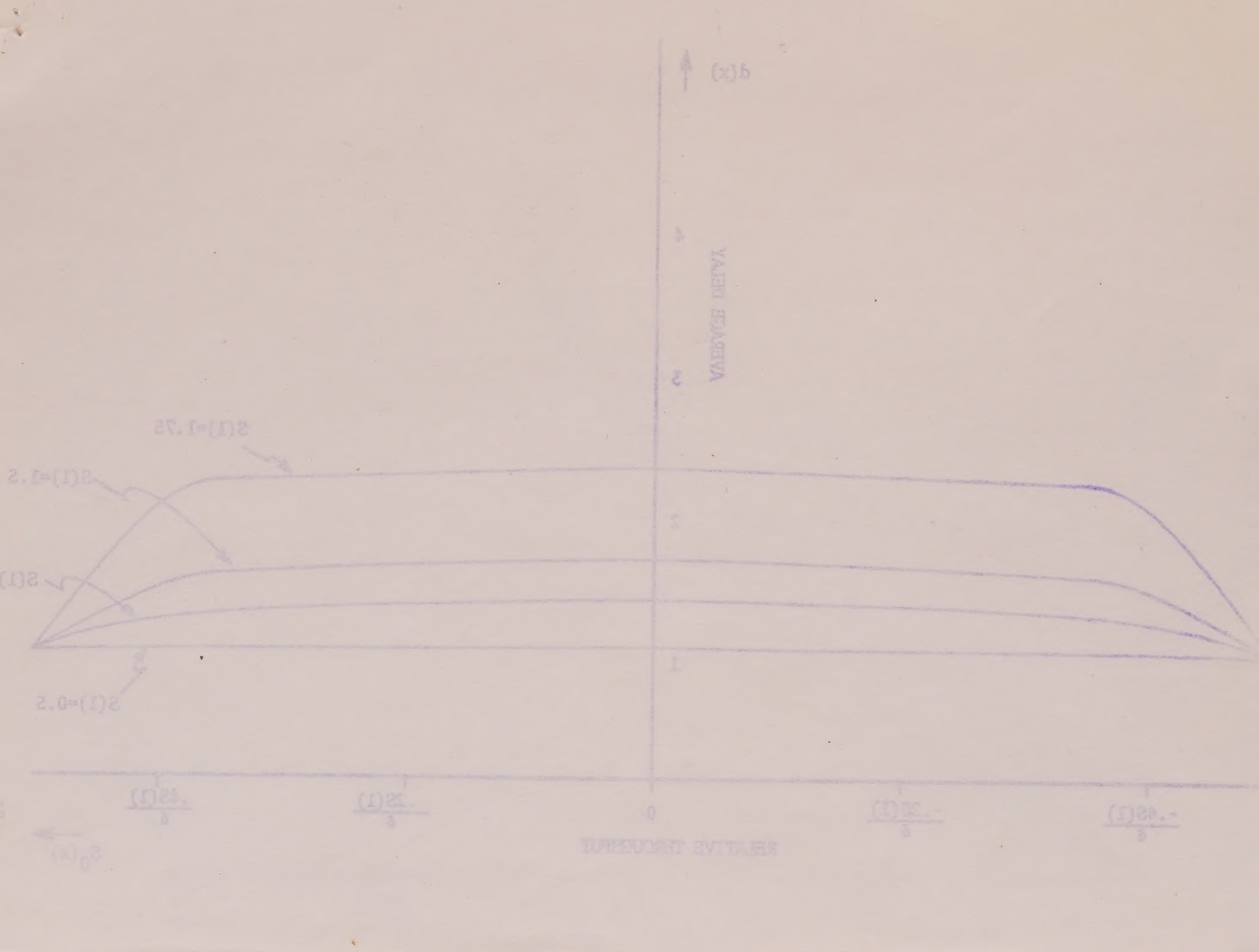



Figure 4 AVERAGE DELAY VS. RELATIVE THROUGHPUT